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Analytic structure of the one-dimensional random-bond Ising model

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Abstract. The quenched average of the one-dimensional random-bond ($\pm J$) Ising model in a magnetic field is studied using a technique based on the transformation of a product of random 2×2 matrices to an iterated conformal map. This approach allows for the underlying analytic structure of the random matrix product to appear as a natural consequence of the conformal mapping of the extended complex plane. The quenched average of the characteristic exponent (i.e. the free energy) is obtained by averaging over a specified probability distribution of nearest-neighbour coupling. The evaluation of the quenched average of the characteristic exponent is performed within the isoentropic approximation. The isoentropic approximation treats all bond configurations with the same number of antiferromagnetic domains having the same entropy and enables the enumeration of equivalent configurations. Combinatorial methods can then be used to express the average over all realizations of the random matrix product as a combinatorial sum. The combinatorial sum is evaluated using resummation methods and an explicit expression for the quenched average of the characteristic exponent is thereby obtained for the case of the random-bond Ising model. The explicit expression for the free energy is dependent on constants which are calculated numerically.

1. Introduction

Products of random 2×2 matrices often appear in the theory of one-dimensional disordered systems. Despite their apparent simplicity, exact solutions exist for relatively few cases and a great deal of effort has been devoted to the study of such systems numerically. The main difficulty arises when the quenched average of thermodynamic quantities are considered. There are several techniques which can be used in the evaluation of the quenched average. The two most frequently used in the context of spin glasses are the replica trick, and the TAP approach [1]. The replica trick consists of averaging over n replicas of the system and then considering the limit $n \to 0$. The TAP approach, first derived by Thouless *et al* is based on averaging the mean-field equations of motion and was formulated to avoid the mathematical difficulties associated with the replica trick. The approach outlined in this work does not use either technique and is based on simple combinatorics to enumerate approximately equivalent configurations of the random-bond Ising model.

Recently, there have been a number of results which give formally exact solutions to the one-dimensional random Ising model. One approach is based on the cycle expansion of the Lyapunov characteristic exponent [2]. This method gives formally exact results using the Ruelle zeta function representation of the generating function for the Ising spin glass. The quenched average is evaluated using the replica trick. The cycle expansion gives numerical

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results for the Lyapunov characteristic exponent to N-digit accuracy using cycles of length N.

An alternative approach has been developed recently which is based on mapping the random matrix product with given site-dependent coupling and field into a new set of couplings and field variables [3]. The mapping is iterated and the fixed-points studied. The fixed-point analysis shows that the mapping converges to a constant coupling fixed-point in zero field. The average is performed by introducing auxiliary spins. This method does not give an explicit expression for the quenched average of the characteristic exponent, but does yield convergent numerical results. This type of iterated mapping is closely related to the approach to be described in the present work.

In the following, an approach based on the transformation of a product of random 2×2 matrices to an iterated conformal mapping is outlined. The approach allows for the underlying analytic structure of the random matrix product to appear as a natural consequence of the conformal mapping of the extended complex plane. Similar bilinear mappings have been obtained [4–6], but have not made use of the conformal structure of the matrix product. The analytic structure and symmetry of spin and matrix models in the complex temperature plane or the complex coupling plane [7, 8] can also be investigated through use of the iterated conformal mapping technique.

The mapping can be parametrized in terms of elementary functions and various limits of the parametrized map are discussed. The quantity studied is the quenched average of the characteristic exponent. The characteristic exponent gives the exponential rate of divergence of the matrix product and the quenched average is taken over a specified probability distribution of near-neighbour coupling. The evaluation of the averaged characteristic exponent is performed using combinatorial methods to express the average over all realizations of the possible configurations of the random matrix product as a combinatorial sum. The combinatorial sum can subsequently be evaluated using resummation methods for combinatorial sums outlined in the monograph by Egorychev [9]. An explicit expression for the quenched average of the characteristic exponent is thereby obtained.

2. The 1D Ising chain

The partition function for the random-bond Ising model (RBIM) is defined as the sum of Boltzmann weights over all configurations of the spins ($\sigma_j = \pm 1$) and is given by

$$Z_N = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp(-\beta H[\sigma]) \tag{1}$$

where

$$-\beta H = \sum_{j=1}^{N} K_j \sigma_j \sigma_{j+1} + h\sigma_j \tag{2}$$

where $K_j = \beta J_j$, J_j is a sequence of site-dependent nearest-neighbour coupling, and h is a constant external magnetic field. For the random-bond Ising model, the sequence of nearest-neighbour coupling are independent random variables with a given probability distribution. The spins (σ_i) are located at the N sites of a one-dimensional lattice subject to periodic boundary conditions. This model defines the simplest spin glass [10] which exhibits non-trivial behaviour due to frustration and has been extensively studied [10, 11].

In the transfer matrix formalism [1, 12], the partition function is the trace of the matrix product given by

$$Z_N = \mathrm{Tr}\left(\prod_{j=1}^{N} T_j\right)$$

where we have used periodic boundary conditions applied to the chain. The transfer matrix takes into account all possible configurations of nearest-neighbour spins and is given by

$$T_j = \begin{pmatrix} \exp(K_j + h)\exp(-K_j) \\ \exp(-K_j)\exp(K_j - h) \end{pmatrix}.$$

When the coupling is site-independent, the partition function is readily evaluated due to the cyclic property of the trace, and is simply the sum of the product of the two eigenvalues ($\lambda_\pm$) of T (i.e. $Z_N = \lambda_+^N + \lambda_-^N$) [12].

For an arbitrary sequence of site-dependent coupling (J_j), the transfer matrices do not, in general, commute and therefore cannot be simultaneously diagonalized. The trace of the matrix product will not be the sum of the product of the eigenvalues. However, the evaluation of the partition function can be obtained by expanding the matrix product in a series and evaluating the trace term by term. To obtain the series expansion, it is useful to express the transfer matrices in terms of the Pauli spin matrices

$$T_j = \mathrm{e}^{K_j}\cosh(h)(I + \boldsymbol{\sigma}\cdot\boldsymbol{a}_j)$$

where I is the 2×2 identity matrix, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices. The vector $\boldsymbol{a}_j$ can be expressed in terms of the site-dependent coupling and external field by

$$\boldsymbol{a}_j = \tanh(h)\hat{\epsilon}_z + \mathrm{e}^{-2K_j}\,\mathrm{sech}(h)\hat{\epsilon}_x \tag{3}$$

where $\hat{\epsilon}_{x,z}$ are unit vectors in along the x- and z-axes, respectively. It is convenient to define

$$\theta_j = \tan^{-1}\left(\frac{\mathrm{e}^{-2K_j}\,\mathrm{sech}(h)}{\tanh(h)}\right) \tag{4}$$

which is the angle relative to the z-axis.

To study the underlying analytic structure, it is necessary to perform a site-independent rotation of the transfer matrices, such that the $\boldsymbol{a}_j$ lies in the x–y plane. This rotation leaves the trace invariant but allows for a complex representation for the transfer matrices. Therefore, the rotated transfer matrices are given by

$$R_j = \begin{pmatrix} 1 & z_j \\ \overline{z_j} & 1 \end{pmatrix}$$

where the z_j are complex numbers given by

$$z_j = \tanh(h) + \mathrm{i}\mathrm{e}^{-2K_j}\,\mathrm{sech}(h) \tag{5}$$

and the phase of z_j is given in (4). The partition function becomes

$$Z_N = \mathrm{e}^{\sum_{j=1}^{N} K_j}\cosh^N(h)\mathrm{Tr}\left(\prod_{j=1}^{N} R_j\right) \tag{6}$$

where the trace is performed on the product of the rotated transfer matrices. The complex form of the rotated transfer matrices is the starting point for the derivation of the iterated conformal map.

2.1. The map

The trace in (6) still cannot be evaluated since the rotated transfer matrices on differing sites, in general, do not commute. However, the rotated transfer matrices are elements of $GL(2,\mathbb{C})$ and these matrices induce a Möbius transformation on the extended complex plane. The matrix product can be expressed as an iterated conformal map which enables the study of the underlying analytic structure of the partition function.

To derive the map, consider the matrices which are elements of $GL(2,\mathbb{C})$ of the form

$$M_j = \begin{pmatrix} q_j & \rho_j \\ \overline{\rho}_j & \overline{q}_j \end{pmatrix}$$

such that $\det(M_j) \neq 1$. It is clear that the rotated transfer matrices are of the same general form. Let us define the partial product of the rotated transfer matrices

$$M_n = \prod_{j=1}^{n} R_j$$

and therefore

$$\mathrm{Tr}(M_N) = 2\ \mathrm{Re}(q_N)\,.$$

Therefore, the partition function is obtained by deriving a recurrence relation for the q_js. This procedure is similar to the one outlined in Derrida and Hilhorst [5] in which a recursion relation for the random-field Ising model is derived. The recurrence relation is derived from

$$M_j = M_{j-1} R_j \tag{7}$$

which follows from the definition of the M_js. Therefore using (7), one obtains a three term recursion relation for

$$q_{j+1} = \left(1 + \frac{\overline{z}_{j+1}}{\overline{z}_j}\right) q_j - \frac{\overline{z}_{j+1}}{\overline{z}_j}(1 - |z_j|^2) q_{j-1} \tag{8}$$

for $2 \leqslant j \leqslant N-1$ with the initial terms, $q_1 = 1$, and $q_2 = 1 + z_1\overline{z}_2$. The trace of the rotated transfer matrices is given by the Nth iterate of (8), and the partition function is given by

$$Z_N = \mathrm{e}^{\sum_{i=1}^{N} K_i} \cosh^N(h)\ 2\ \mathrm{Re}(q_N)\,. \tag{9}$$

For the case of $J_j = J$ constant coupling, (8) reduces to a finite difference equation with constant coefficients and can be evaluated to give the standard transfer matrix result.

The recurrence relation can be simplified by considering the ratio $X_j = q_j/q_{j-1}$. This gives rise to a tail sequence, and the Nth iterate is given by

$$q_N = \prod_{j=1}^{N} X_j\,.$$

The recurrence for X_j becomes

$$X_{j+1} - 1 = \frac{\overline{z}_{j+1}}{\overline{z}_j}\left(1 - \frac{1 - |z_j|^2}{X_j}\right)$$

with initial terms, $X_2 = 1 + z_1\overline{z}_2$, and $X_1 = 1$. Further simplification is possible through the introduction of a new rescaled variable $w_j = (X_j - 1)/\overline{z}_j$. We obtain the simple map

$$w_{j+1} = \frac{w_j + z_j}{1 + \overline{z}_j w_j} \tag{10}$$

with the initial term $w_1 = 0$. The resulting expression for the partition function is given by

$$Z_N = e^{\sum_{i=1}^{N} K_i} \cosh^N(h)\ 2\ \mathrm{Re}\left(\prod_{i=1}^{N} 1 + \bar{z}_i w_i\right) \tag{11}$$

in terms of the bilinear mapping derived in (10).

Equation (10) for w_{j+1} is of the form of a Möbius transformation of the extended complex plane. This type of bilinear mapping of the extended complex plane is conformal (angle preserving). and has the properties that it maps circles onto circles and lines onto lines, and for $|z| < 1$ it maps $|w| < 1$ into the unit circle ($|w'| < 1$) [13, 14]. The fixed-points of the Möbius transformation are given by $w_{\mathrm{fp}} = \pm e^{i\theta_j}$ and are symmetric about the origin and lie on the unit circle. Furthermore, for w_j on the unit circle it is easy to show that w_{j+1} remains on the unit circle. Thus the map takes the unit circle in the complex plane onto the unit circle. This is a property of the underlying conformal structure of the map and will be used in the evaluation of the partition function.

The input sequence z_j for some general J_j lie along the line $\mathrm{Re}(z_j) = \tanh(h)$ in the complex plane. For ferromagnetic coupling $J_j > 0$, the corresponding z_j lie along the line within the unit circle. For antiferromagnetic coupling $J_j < 0$, the corresponding z_j lie along the line outside the unit circle. For $J_j = 0$, this corresponds to $|z_j| = 1$ which results in a singular Möbius transformation. In the limit of infinite ferromagnetic coupling, z_j approaches the real axis along the line $\mathrm{Re}(z_j) = \tanh(h)$. In the limit of infinite antiferromagnetic coupling, z_j diverges along the line $\mathrm{Re}(z_j) = \tanh(h)$.

2.2. *Parametrization of the map*

The Möbius transformation can be parametrized in terms of elementary functions with the remarkable feature that the parametrization is valid in both the ferromagnetic ($|z_j| < 1$) and the antiferromagnetic ($|z_j| > 1$) regions of the complex plane. To facilitate the parametrization, we define a new rescaled variable $u_j = e^{-i\theta_j} w_j$. The transformed map reads

$$u_{j+1} = e^{-i(\theta_{j+1} - \theta_j)} \left(\frac{u_j + |z_j|}{1 + |z_j| u_j}\right). \tag{12}$$

Equation (12) is parametrized using the following definitions:

$$u_j = \tanh(\zeta_j) \qquad |z_j| = \tanh(\phi_j)$$

with

$$\phi_j = \tfrac{1}{2}\log\left(\frac{1 + |z_j|}{1 - |z_j|}\right)$$

where ϕ_j is complex in the antiferromagnetic region. From the above definitions, we obtain the parametrized conformal map of (12) given by

$$\tanh(\zeta_{j+1}) = e^{-i(\theta_{j+1} - \theta_j)} \tanh(\zeta_j + \phi_j) \tag{13}$$

for $1 \leqslant j \leqslant N - 1$ with initial term $\zeta_1 = 0$.

Equation (13) in parametrized form is valid in both regions of the complex plane and is nearly additive in the parameter ζ_j . The presence of the phase destroys the additivity and results from the non-commutativity of the rotated transfer matrices. In the limit of vanishing field $h \to 0$, the phase approaches $\pi/2$ and the phase difference vanishes. Thus, in zero field the additive solution is given by

$$\zeta_n = \sum_{i=1}^{n-1} \phi_i \tag{14}$$

which corresponds to the product of the eigenvalues of the rotated transfer matrices. In a weak field, the parametrization is approximately additive and an approximate form for the partition function can be obtained.

In terms of the parametrized mapping given by (13), the complex eigenvalues of the product of the rotated transfer matrices are given by

$$\Lambda^{(+)} = \prod_{j=1}^{N} \frac{\cosh(\zeta_j + \phi_j)}{\cosh(\phi_j)\cosh(\zeta_j)} \tag{15}$$

and $\Lambda^{(-)}$ is the complex conjugate of $\Lambda^{(+)}$, and the partition function is

$$Z_N = \mathrm{e}^{\sum_{i=1}^{N} K_i} \cosh^N(h) \left(\Lambda^{(+)} + \Lambda^{(-)}\right). \tag{16}$$

We want to investigate the deviation from the commutative case by using the additive solution of (13), and defining new map parameters

$$\zeta_n = \sum_{i=1}^{n-1} \phi_i + \nu_n \tag{17}$$

where the first term is the additive result and the second term contains the non-commutative part of the mapping. It is clear from the explicit form of the parametrized map that for vanishing phase difference $\theta_{j+1} - \theta_j \to 0$, we require that $\nu_{j+1} = \nu_j$.

To examine the effect of the separation of the commutative and non-commutative map parametrization on $\Lambda^{(+)}$, we introduce relative map parameters, $\Delta_n = \nu_{n+1} - \nu_n$, and $\Gamma_n = \nu_{n+1} + \nu_n$ rather than work with the set of non-commutative parameters, $\{\nu\}$. The non-commutative parameters can be expressed in terms of partial sums of the relative map parameters. Explicitly, we find that

$$\Gamma_n = \Delta_n + 2\sum_{i=1}^{n-1} \Delta_i \,. \tag{18}$$

The motivation for the redefined map parameters can be seen within the context of the $\pm J$ model. Consider a domain of ferromagnetic (J) or antiferromagnetic ($-J$) coupling. The non-commutative map parameter does not change within a given domain and changes discontinuously at a domain boundary. This can be readily seen from (13). Therefore, Δ_n vanishes within a domain and is non-zero at domain boundary.

Equation (18) can be used to rewrite $\Lambda^{(+)}$ as the product of two terms

$$\Lambda^{(+)} = \Lambda_a \Lambda_b \tag{19}$$

where

$$\Lambda_a = \prod_{i=1}^{N} (1 + |z_i|) + \mathrm{e}^{-2\sum_{i=1}^{N-1} \Delta_i} \prod_{i=1}^{N} (1 - |z_i|) \tag{20}$$

and

$$\Lambda_b = \prod_{n=1}^{N-1} \left(\frac{1 + Q_n \mathrm{e}^{-2\sum_{i=1}^{n-1} \Delta_i}}{1 + Q_n \mathrm{e}^{-2\sum_{i=1}^{n} \Delta_i}} \right) \tag{21}$$

where

$$Q_n = \prod_{i=1}^{n} \left(\frac{1 - |z_i|}{1 + |z_i|} \right). \tag{22}$$

It is important to note that the zero field (additive result) is recovered since in the zero field case all the $\Delta_n = 0$. and we obtain the product of the eigenvalues of the transfer matrices.

In (20), the first term in Λ_a is the product of the large eigenvalues of the rotated transfer matrices, and the second term is the weighted product of the small eigenvalues of the rotated transfer matrices. The non-commutativity enters through the weighting factor of the small eigenvalue and depends on the domain structure. Furthermore, Q_n is the partial product of the ratio of the small to the large eigenvalue of the transfer matrix and is less than unity, therefore it is expected that $\Lambda_b \approx 1$ in (19), and Λ_a will be the dominant contribution.

2.3. A simple example

Let us consider the case when the coupling takes on two values $J_i = \pm J$. Explicitly, we consider k antiferromagnetic bonds chosen at random out of the total of N bonds of the chain. These antiferromagnetic impurities will form q domains. Therefore, for a chain of length N we will have Nx antiferromagnetic bonds forming q domains, where x is the concentration of the antiferromagnetic impurities. For any fixed configuration of the k antiferromagnetic impurities, we must evaluate (13) to determine $\Lambda_k^{(+)}$ and hence the partition function.

For $K_i = K$, we define $|z_i| = u$, and for $K_i = -K$ $|z_i| = v$. From (13), it is clear that the only non-zero contributions for v occur at a domain boundary since the phase difference does not vanish at a domain boundary. Therefore, the Δ_js are zero except at a domain wall, which implies that $\sum_{i=1}^{N-1} \Delta_i = 2q\Delta$ is the average of the difference of the non-commutative map parameters. Here Δ is a complex quantity determined numerically, which depends on the parameter values chosen, and $2q$ is the number of domain walls on the chain.

In figure 1, we show the results for $\mathrm{Re}(v)$ versus iteration number. The gaps in the plot occur at the domain walls where the non-commutative map parameter changes discontinuously. From figure 1, an estimate of $\mathrm{Re}(\Delta)$ can be obtained by computing the average slope of the graph. Figure 2 shows the imaginary part of Δ, which after only a few iterations converges to three possible values, $\mathrm{Im}(\Delta) = 0, \pi/2, \pi$ and enables the exact determination of the imaginary part of Δ. Furthermore, numerical simulation indicates that after only approximately 10 iterations, $|\tanh(\zeta)| = 1$. Figure (3) indicates the convergence of the map to the unit circle. This is a result of the general properties of the underlying conformal map, which has attractive fixed-points lying on the unit circle.

The approximation for $\mathrm{Re}(\Delta)$ can be used in the calculation of $\Lambda^{(+)}$. To find an approximate solution for $\Lambda^{(+)}$ with k antiferromagnetic impurities forming q domains, the results of the previous section are used ($\Lambda_b \approx 1$ for large N) and we obtain

$$\Lambda_k^{(+)}(q) \approx (1+u)^{N-k}(1+v)^k + \mathrm{e}^{-4q\Delta}(1-u)^{N-k}(1-v)^k \tag{23}$$

which is valid in the limit of large N. For $\Delta = 0$, (23) reduces to the commutative result and we have therefore obtained an approximate form for complex eigenvalue for the rotated transfer matrices, which depends on the number of domains q and the complex parameter Δ. Furthermore, from the values of $\mathrm{Im}(\Delta)$ obtained in figure (2), it is easily shown that the approximate expression for $\Lambda_k^{(+)}$ is real.

3. Random-bond model

We want to consider the case when the bonds are distributed according to the simple probability distribution given by

$$\rho = \prod_{i=1}^{N} p\delta(J - J_i) + (1-p)\delta(J + J_i) \tag{24}$$

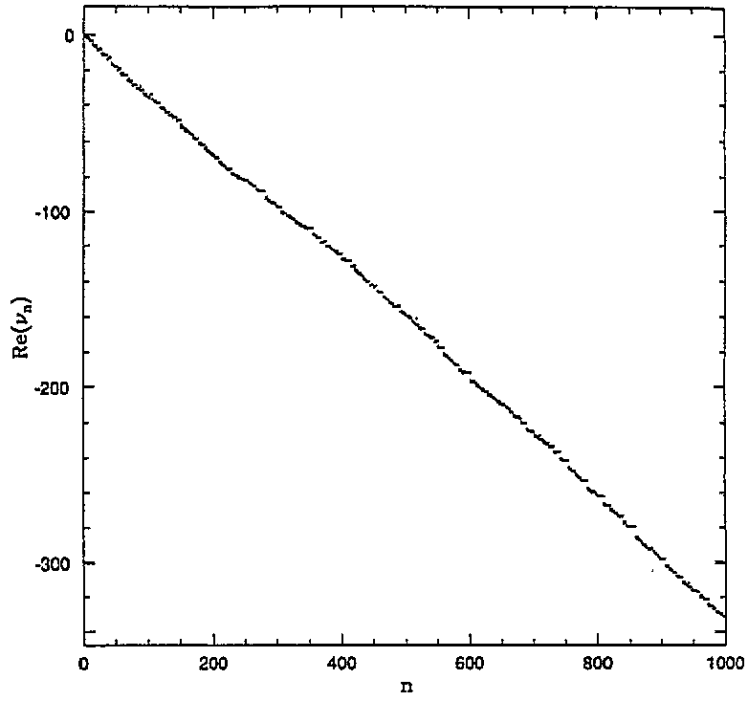

Figure 1. The real part of ν versus iteration number for $x = \frac{1}{3}$, $|K| = 1.5$ and $h = 0.5$. The number of domains is $q = 233$ and $\mathrm{Re}(\Delta) = -0.71$.

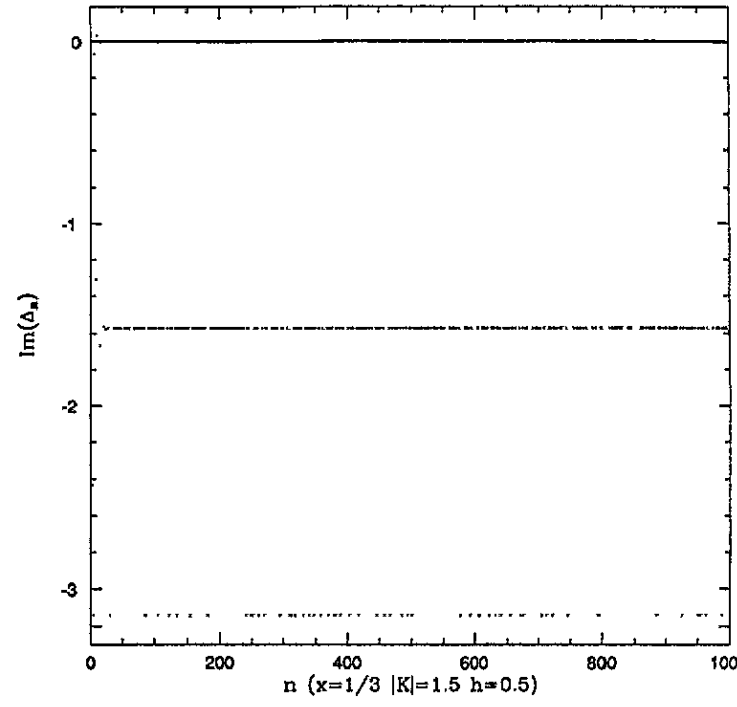

Figure 2. Imaginary part of Δ versus iteration number for $x = \frac{1}{3}$, $|K| = 1.5$ and $h = 0.5$.

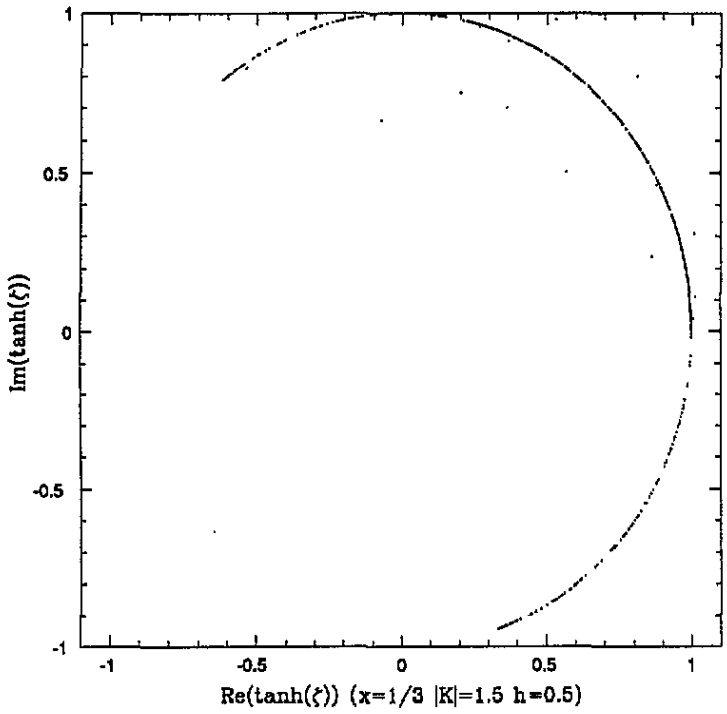

Figure 3. Imaginary part of $\tanh(\zeta)$ versus the real part for $x = \frac{1}{3}$, $|K| = 1.5$ and $h = 0.5$.

where p is the probability that a given site has ferromagnetic coupling, and $1 - p$ is the probability that a site has antiferromagnetic coupling. It is convenient to expand the distribution in a binomial expansion to obtain

$$\rho = \sum_{k=0}^{N} p^{N-k}(1-p)^k \rho_k \tag{25}$$

where

$$\rho_k = \sum_{l_1<l_2}^{N} \cdots \sum_{<l_k}^{N} \prod_{i=1}^{k} \delta(J + J_{l_i}) \prod_{j \neq \{l\}} \delta(J - J_j)\,. \tag{26}$$

This type of series expansion of the probability distribution will lead to a combinatorial sum when averages over the expanded distribution are computed.

The quantity of interest is the quenched average of the characteristic exponent where the characteristic exponent is defined as

$$\gamma_N = \frac{\log(\Lambda^+)}{N}\,. \tag{27}$$

The quenched average of the characteristic exponent is calculated by using the binomial expanded version of the distribution and is given by

$$\langle \gamma_N \rangle = \frac{1}{N} \sum_{k=0}^{N} p^{N-k}(1-p)^k \int \mathrm{D}J \rho_k \log\left(\Lambda^{(+)}\right) \tag{28}$$

where $\mathrm{D}J = \prod_{i=1}^{N} \mathrm{d}J_i$.

The quenched average of the characteristic exponent can be expressed as a combinatorial sum of impurity configurations of the one-dimensional chain. The integral in (28) is over all values of the coupling J_i which picks out all the configurations with k antiferromagnetic impurities and sums over all possible realizations of these configurations.

In general, (28) depends on the position of the k impurities through the distance separating domains. This dependence can be viewed as an effective interaction between domains. By neglecting the implicit distance dependence in $\Lambda_k^{(+)}$, the domains can be considered as non-interacting and configurations with the same number of domain walls are equivalent. This approximation of non-interacting domains is called the isoentropic approximation since the entropy of all configurations with the same number of domain walls is equal.

This approach allows for the use of simple counting techniques to enumerate the different configurations of impurities, and we use the approximate expression for $\Lambda_k^{(+)} \to \Lambda_k^{(+)}(q)$ derived in the previous section in which k impurities forming q domains was considered. This will enable the evaluation of the combinatoric sum for the quenched average of the characteristic exponent within the isoentropic approximation. The sum over the configurations in (28) for the characteristic exponent is thus simplified and is given by

$$\langle \gamma_N \rangle = \frac{1}{N} \sum_{k=0}^{N} \frac{p^{N-k}(1-p)^k}{k!} \sum_{q=0}^{k} g_N(k,q) \log(\Lambda_k^{(+)}(q)) \tag{29}$$

where $g_N(k,q)$ counts the number of ways of choosing k antiferromagnetic bonds out of N possible bonds such that they form q domains. The combinatorial factor is evaluated using the generating function technique, and is found to be

$$g_N(k,q) = N(k-1)! \begin{pmatrix} N-k-1 \\ q-1 \end{pmatrix} \begin{pmatrix} k \\ q \end{pmatrix}.$$

The resulting combinatoric sum over both k and q indices can be evaluated using a symbolic residue calculus for the binomial coefficients (see [9]). This calculation is outlined in the appendix.

The quenched average of the characteristic exponent is obtained by expanding the logarithm in the combinatorial sum. This is a formal expansion of the logarithm and we will consider the two possible factorizations of $\Lambda_k^{(+)}(q)$. We can factor out the first term in $\Lambda_k^{(+)}(q)$ and the quenched average of the characteristic exponent is given by

$$\langle \gamma_N^{(+)} \rangle = p\log(1+u) + (1-p)\log(1+v) + \frac{p^N}{N}\log\left(1+a^N\right) - \frac{(1-p)^N}{N}\log\left(1+b^N\right) + \overline{\gamma} \tag{30}$$

within the isoentropic approximation. We define

$$\overline{\gamma} = \frac{1}{(2p)^N} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left(\lambda_1^N + \lambda_2^N\right) \tag{31}$$

where

$$\lambda_1 = pa^r + (1-p)b^r + \sqrt{4(1-p)b^r e^{-4r\Delta} + (pa^r - (1-p)b^r)^2} \tag{32}$$

and

$$\lambda_2 = pa^r + (1-p)b^r - \sqrt{4(1-p)b^r e^{-4r\Delta} + (pa^r - (1-p)b^r)^2}. \tag{33}$$

We also define the ratio of the eigenvalues of the rotated transfer matrices for the ferromagnetic case as

$$a = \frac{1-u}{1+u}$$

and for the antiferromagnetic case as

$$b = \frac{1-v}{1+v}.$$

Alternatively, we can factor out the second term in the expression for $\Lambda_k^{(+)}(q)$ and the quenched average of the characteristic exponent becomes

$$\langle \gamma_N^{(-)} \rangle = p\log(1-u) + (1-p)\log(|1-v|) + i\pi(1-p) + \frac{p^N}{N}\log(1+a^N) - \frac{(1-p)^N}{N}\log(1+b^N) - 4\Delta(1-p)\left(\frac{p^N-(1-p)^N}{2p-1}\right) + \overline{\gamma} \tag{34}$$

where $\overline{\gamma}$ was defined previously. We note that

$$\lambda_1 = pa^r + (1-p)b^r + \sqrt{4(1-p)b^r e^{4r\Delta} + (pa^r - (1-p)b^r)^2} \tag{35}$$

and

$$\lambda_2 = pa^r + (1-p)b^r - \sqrt{4(1-p)b^r e^{4r\Delta} + (pa^r - (1-p)b^r)^2}. \tag{36}$$

Furthermore, we define the ratio of the eigenvalues of the rotated transfer matrices for the ferromagnetic case as

$$a = \frac{1+u}{1-u}$$

and for the antiferromagnetic case as

$$b = \frac{1+v}{1-v}.$$

Equations (30) and (34) represent the quenched average of the characteristic exponent of the $\pm J$ spin glass within the isoentropic approximation for a finite lattice of N sites. The quantities a and b are simply the ratios of the eigenvalues of the transfer matrices. In general, the ratio of the antiferromagnetic eigenvalues are negative ($b < 0$) which implies that λ_1 and λ_2 can be complex quantities which are mutually conjugate. Therefore, it is expected that $\overline{\gamma}$ is real. Furthermore, in the expressions for λ_1 and λ_2, we note that the ratio of the ferromagnetic and antiferromagnetic eigenvalues raised to the power r are weighted by the probability of ferromagnetic coupling and antiferromagnetic coupling, respectively. The expression for $\overline{\gamma}$ treats the two types of coupling in a symmetric fashion.

First, we evaluate $\langle \gamma_N^{(-)} \rangle$ since λ_1 and λ_2 are real valued. In figure 4 the ratio $|\lambda_2/\lambda_1|$ versus r is shown; the inset shows the plot of $\log(\lambda_1)$ versus r. From these two figures, we infer that $\lambda_2 \ll \lambda_1$ and that λ_1 is an exponential function of r of the form $\lambda_1 = C\exp(\alpha r)$. The quantities C and α can be obtained numerically and we can approximate

$$\overline{\gamma} \simeq \frac{C^N}{(2p)^N}\log(1+e^{\alpha N}).$$

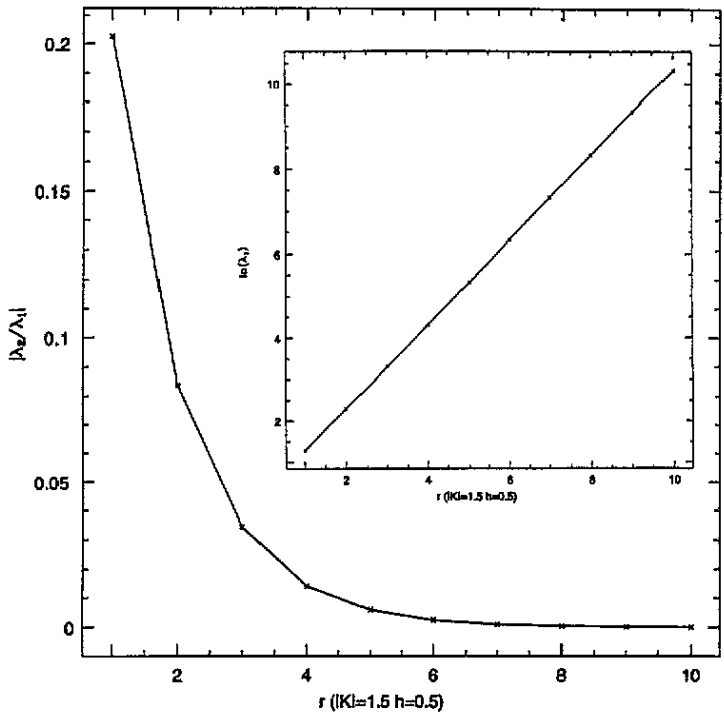

Figure 4. Ratio of $|\lambda_2/\lambda_1|$ versus r, $x = \frac{1}{3}$, $|K| = 1.5$ and $h = 0.5$. The inset is a plot of $\log(\lambda_1)$ versus r.

From the above expression we obtain

$$\langle \gamma_N^{(-)} \rangle \simeq p\log(1-u) + (1-p)\log(|1-v|) + i\pi(1-p) + \frac{p^N}{N}\log\left(1+a^N\right) - \frac{(1-p)^N}{N}\log\left(1+b^N\right) - 4\Delta(1-p)\left(\frac{p^N-(1-p)^N}{2p-1}\right) + \frac{C^N}{(2p)^N}\log\left(1+e^{\alpha N}\right) \tag{37}$$

where a and b are the ratios of the large to the small eigenvalues of the transfer matrices.

Alternatively, we can evaluate $\langle \gamma_N^{(+)} \rangle$ where λ_1 and λ_2 are mutually conjugate. In figure (5) $\log(|\lambda_1|)$ versus r is shown. From this figure, it is clear that modulus of λ_1 is an exponential function of r of the form $|\lambda_1| = C' \exp(\beta r)$. The phase of λ_1 can be determined numerically and is zero for r even, and $\pi/2$ for r odd. The quantities C' and β can likewise be obtained numerically and we can approximate

$$\overline{\gamma} \simeq \frac{2C'^N}{(2p)^N}\left(\cos^2\left(\frac{\pi N}{4}\right)\log\left(1+e^{\beta N}\right) + \sin^2\left(\frac{\pi N}{4}\right)\log\left(1-e^{\beta N}\right)\right).$$

From the above expression we obtain

$$\langle \gamma_N^{(+)} \rangle \simeq p\log(1+u) + (1-p)\log(1+v) + \frac{p^N}{N}\log\left(1+a^N\right) - \frac{(1-p)^N}{N}\log\left(1+b^N\right) + \frac{2C'^N}{(2p)^N}\left(\cos^2\left(\frac{\pi N}{4}\right)\log\left(1+e^{\beta N}\right) + \sin^2\left(\frac{\pi N}{4}\right)\log\left(1-e^{\beta N}\right)\right) \tag{38}$$

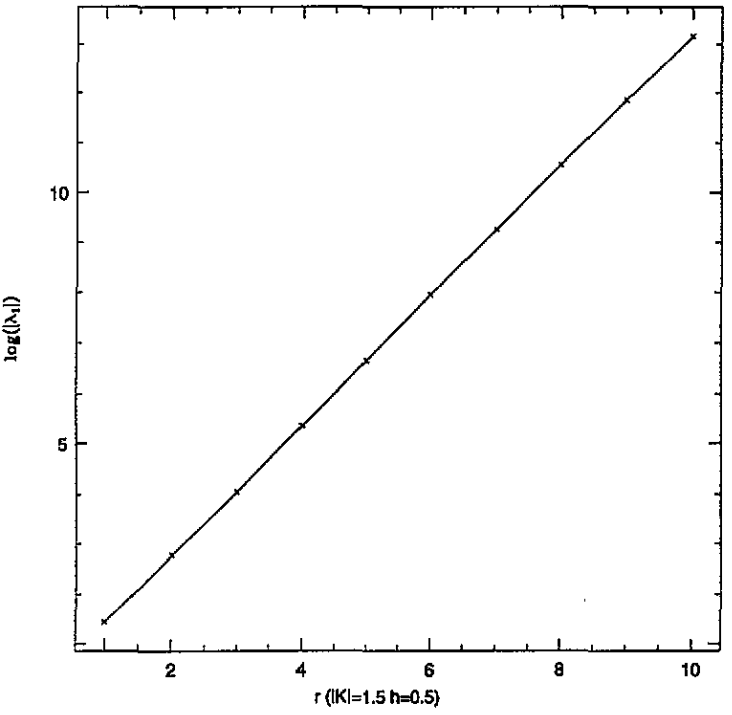

Figure 5. Plot of $\log(|\lambda_1|)$ versus r for $x = \frac{1}{3}$, $|K| = 1.5$ and $h = 0.5$.

where a and b are the ratios of the small to the large eigenvalues of the transfer matrices.

Equations (37) and (38) represent approximate expressions for the quenched average of the characteristic exponent for both factorizations. They depend on Δ, C and α or C' and β which are determined from the numerical results.

4. Conclusions

We have demonstrated that the random matrix product can be represented by an iterated conformal map. The underlying analytic structure of the matrix product can be understood from the general properties of the conformal mapping. The fixed points of the conformal map lie on the unit circle and we have demonstrated that for $\pm J$ model that the mapping converges rapidly to the unit circle.

Furthermore, the complex eigenvalues and the map itself can be parametrized in terms of elementary functions and the case of non-commutative transfer matrices is manifest as a simple phase in the parametrized map. The parametrized map can also be used to derive an approximate expression for the complex eigenvalues of the matrix product. The approximate form of the complex eigenvalues can be used in the evaluation of the quenched average of the characteristic exponent in the $\pm J$.

The quenched average of the characteristic exponent is then evaluated by simple counting arguments to enumerate the various impurity configurations. It is important to note that the replica trick has not been employed to evaluate the averages and that by using resummation techniques, we are able to obtain an explicit expression for the quenched average of the characteristic exponent which depends on a small set of numerically

determined parameters.

Finally, the technique of conversion of matrix products to iterated conformal maps is general and can be applied to other one-dimensional disordered systems.

The investigation of the one-dimensional Anderson model with binary on site disorder is amenable to this approach [15, 16].

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Appendix

The evaluation of the combinatoric sum is performed using a symbolic residue calculus for the binomial coefficients. In other words the binomial coefficients are represented as contour integrals

$$\left(\begin{array}{c} N \\ k \end{array} \right) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{(1+z)^N}{z^{k+1}} .$$

Using the above expression for the binomial coefficient, the combinatorial sum in (29) is given by

$$\langle \gamma \rangle = \frac{p^N}{N} \left(\log(1+a^N) + N \sum_{k=1}^{N} \frac{v^k}{k} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} (1+z)^{N-k-1} \sum_{q=1}^{k} \left(\begin{array}{c} k \\ q \end{array} \right) \log\left(1 + \mathrm{e}^{\pm 2q\Delta} a^{N-k} b^k\right) \right)$$

where $v = (1-p)/p$ and a and b are the ratios of the eigenvalues as defined in the text. It is important to note that we have already factored out the first or the second term in the approximate expression for $\Lambda_k^{(+)}(q)$.

The sum over q can be evaluated by expanding the logarithm and using the binomial expansion and is $(1 + w/z)^k - 1$ where $w = \mathrm{e}^{\pm 4r\Delta}$ and r is an integer index from the expansion of the log. We now want to evaluate the sum over k in the second term in the above expansion. This can be done by using the series expansion for $\log(1 + x)$. Concentrating on the second term, we find

$$C = Np^N \sum_{r=1}^{\infty} (-)^{r+1} \frac{a^{Nr}}{r} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} (1+z)^{N-1} \log\left(\frac{z(1+z) - zx}{z(1+z) - x(z+w)} \right)$$

where

$$x = \frac{(1-p)}{p} \left(\frac{b^r}{a^r} \right)$$

and both x and w depend on the index r.

To proceed, we now must evaluate the contour integral over a contour which excludes the origin since the integrand has an essential singularity at $z = 0$. This is done by noting that

$$N(1+z)^{N-1} \log\left(\frac{z(1+z) - zx}{z(1+z) - x(z+w)} \right) = \frac{\mathrm{d}}{\mathrm{d}z} (1+z)^N \log\left(\frac{z(1+z) - zx}{z(1+z) - x(z+w)} \right)$$
$$+ (1+z)^N \left(\frac{2z+1-x}{z(1+z) - zx} - \frac{2z+1-x}{z(z+1) - x(z+w)} \right) .$$

The contour integral can be computed since the second term on the RHS has simple poles at $z_0 = -1 + x$, and at

$$z_\pm = -\frac{(1-x)}{2} \pm \frac{\sqrt{(1-x)^2 + 4xw}}{2}.$$

The total derivative gives no contribution to the contour integral, and the pole at $z = 0$ gives no contribution since the contour is chosen in such a way that the origin is excluded. The only non-zero contribution comes from the residues at $z_\pm$ and z_0. Using the symbolic residue theorem, the combinatoric sum is

$$\langle \gamma_N \rangle = \frac{p^N}{N}\log(1 + a^N) - \frac{(1-p)^N}{N}\log(1 + b^N) + \overline{\gamma}$$

and we find

$$\overline{\gamma} = \sum_{r=1}^{\infty}(-)^{r+1}\frac{a^{rN}}{r}\left((1+z_+)^N + (1+z_-)^N\right)$$

which is the result obtained in (31).

References

[1] Binder K and Young A P 1986 *Rev. Mod. Phys.* **58** 801
[2] Mainieri R 1992 *Phys. Rev. Lett.* **68** 1965
[3] Paladin G and Serva M 1992 *Phys. Rev. Lett.* **69** 706
[4] Dyson F J 1953 *Phys. Rev.* **92** 1331
[5] Derrida B and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 2641
[6] Itzykson C and Drouffe J 1989 *Statistical Field Theory* vol 2 (Cambridge: Cambridge University Press)
[7] Damgaard P H and Heller U M 1993 *Nucl. Phys.* B [FS] **410** 494
[8] Marchesini G and Shrock R E 1989 *Nucl. Phys.* B **318** 541
[9] Egorychev G P 1984 Integral representation and computation of combinatorial sums *Translation of Mathematical Monographs 59* (American Mathematical Society)
[10] Derrida B Vannimenus J and Pomeau Y 1978 *J. Phys. C: Solid State Phys.* **11** 4749
[11] Gardner E and Derrida B 1985 *J. Stat. Phys.* **39** 367
[12] Huang K 1987 *Statistical Mechanics* 2nd edn (New York: Wiley)
[13] Beardon A F 1991 *Iteration of Rational Functions* (New York: Springer)
[14] Nehari Z 1991 *Conformal Mapping* (New York: Dover)
[15] Stone A D, Joannopoulos J D and Chadi D J 1981 *Phys. Rev.* B **24** 5583
[16] Evangelou S N and Economou E N 1993 *J. Phys. A: Math. Gen.* **26** 2803